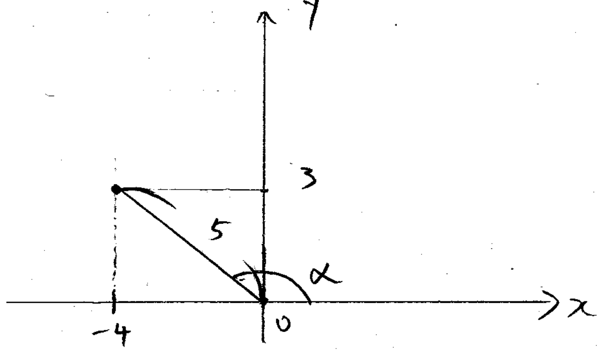
問題 1

(1)

α が第2象限の角で, $\cos\alpha = -\frac{4}{5}$ から



$\sin\alpha = \frac{3}{5}$, $\tan\alpha = -\frac{3}{4}$ となり

$\sin^2\frac{\alpha}{2}$ は、加法定理より

$$\cos\left(\frac{\alpha}{2}-\frac{\alpha}{2}\right) = \cos\frac{\alpha}{2}\cos\frac{\alpha}{2} + \sin\frac{\alpha}{2}\sin\frac{\alpha}{2}$$
$$= \cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}$$
$$\cos\left(\frac{\alpha}{2}+\frac{\alpha}{2}\right) = \cos\frac{\alpha}{2}\cos\frac{\alpha}{2} - \sin\frac{\alpha}{2}\sin\frac{\alpha}{2}$$
$$= \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}$$

$$\cos 0 - \cos\alpha = 2\sin^2\frac{\alpha}{2}$$

$$\sin^2\frac{\alpha}{2} = \frac{1}{2}(1-\cos\alpha)$$
$$= \frac{1}{2}\left(1+\frac{4}{5}\right) = \frac{9}{10}$$

問題1

(2) 座標 $(-2,3)$, $(1,6)$ をそれぞれ

$y = ax^2+bx+1$ に代入して

$$\begin{cases} 3 = 4a-2b+1 & \cdots ① \\ 6 = a+b+1 & \cdots ② \end{cases}$$

① + 2×②

$$\begin{array}{rl} & 4a-2b = 2 \\ +) & 2a+2b = 10 \\ \hline & 6a \quad = 12 \end{array}$$

$$\underline{a = 2}$$

②に代入、

$2+b=5$

$$\underline{b=3}$$

a、b を代入して、平方完成

$$\begin{aligned} y &= 2x^2+3x+1 \\ &= 2\left(x^2+\frac{3}{2}x\right)+1 \\ &= 2\left\{\left(x+\frac{3}{4}\right)^2-\frac{9}{16}\right\}+1 \\ &= 2\left(x+\frac{3}{4}\right)^2-\frac{9}{8}+1 \\ &= 2\left(x+\frac{3}{4}\right)^2-\frac{1}{8} \end{aligned}$$

よって頂点の座標は

$$\left(-\frac{3}{4},\ -\frac{1}{8}\right)$$

増減表による確認

$y = 2x^2 + 3x + 1$

$y' = 4x + 3$

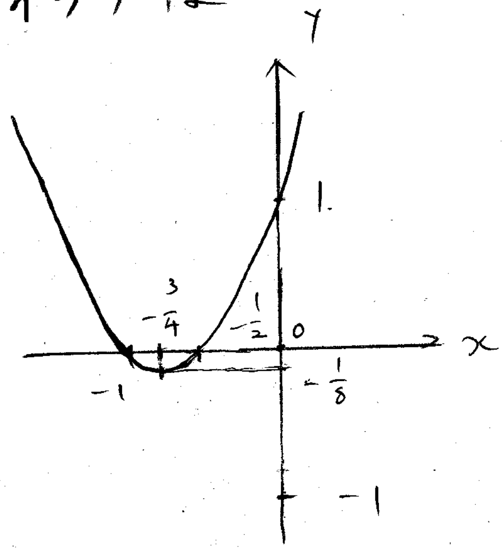
$y' = 0$ のときの x は

$4x + 3 = 0$

$x = -\frac{3}{4}$ なので

x	$\cdots$	$-\frac{3}{4}$	$\cdots$
y'	$-$		$+$
y	$\searrow$	$-\frac{1}{8}$	$\nearrow$

グラフは



問2

(1)

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6 & -2+2 \\ -3+3 & 6+1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{pmatrix} 1 & -3 \\ -2 & -1 \end{pmatrix}$$

$$|A| = \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix}$$

$$= -1-6$$

$$= -7 \quad \text{より}$$

$$A^{-1} = -\frac{1}{7}\begin{pmatrix} 1 & -3 \\ -2 & -1 \end{pmatrix}$$

問2

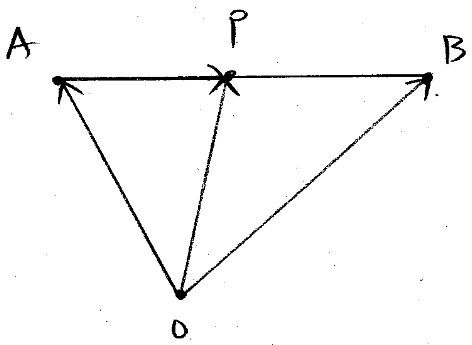
(2)

$A(3, -1, 2)$, $B(4, -1, 1)$

$$\overrightarrow{AB} = (4-3)\vec{i} + (-1+1)\vec{j} + (1-2)\vec{k}$$

$$= \vec{i} - \vec{k}$$

A, Bを通るベクトル方程式は



$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

今、$\overrightarrow{AP} \parallel \overrightarrow{AB}$ より

$\overrightarrow{AP} = t\,\overrightarrow{AB}$　（tは任意の定数）

$$\overrightarrow{OP} = \overrightarrow{OA} + t\,\overrightarrow{AB}$$

$$= 3\vec{i} - \vec{j} + 2\vec{k} + t(\vec{i} - \vec{k})$$

$$= (3+t)\vec{i} - \vec{j} + (2-t)\vec{k}$$

これより

A, Bを通る方程式は

$$\begin{cases} x = 3+t \\ y = -1 \\ z = 2-t \end{cases}$$

$t = x-3$

$t = -z+2$

$x-3 = -z+2$

$y = -1$

問3

(1) $y = x^3 - 3x + 2 \quad (0 \leq x \leq 2)$

$y' = 3x^2 - 3$

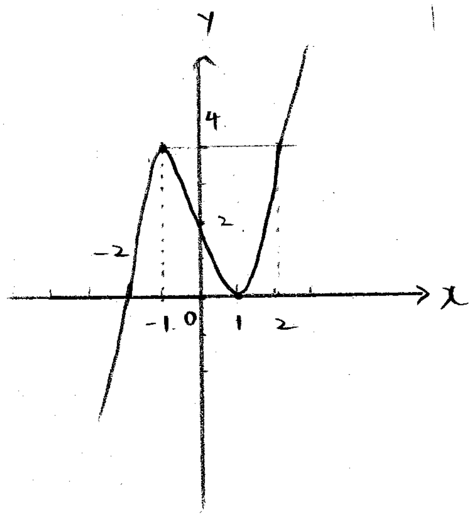
$y' = 0$ のとき x は

$3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$,

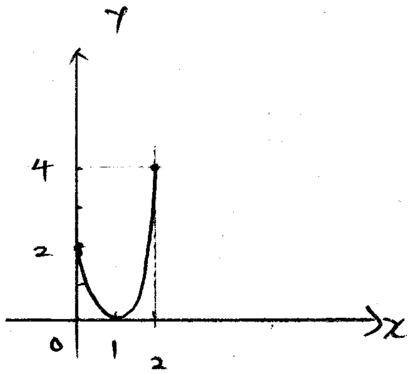
x	$\cdots$	-1	$\cdots$	1	$\cdots$
y'	$+$	×	$-$	×	$+$
y	↗	4	↘	0	↗



$x = -1$ のとき
極大値 $y = 4$ をとる

$x = 1$ のとき
極小値 $y = 0$ をとる

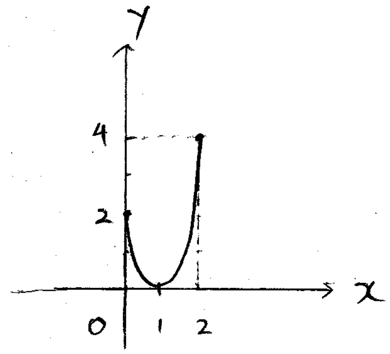
また $0 \leq x \leq 2$ のとき



問3

(2)

(1)より



$x=1$ のとき

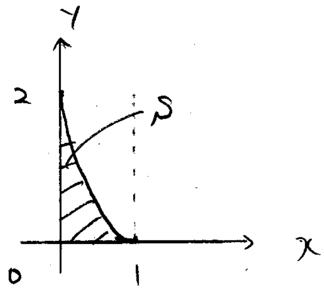
最小値 $y=0$

$x=2$ のとき

最大値 $y=4$

(3)

(1)より



$$S=\int_0^1 x^3-3x+2\,dx$$

$$=\left[\frac{1}{4}x^4-\frac{3}{2}x^2+2x\right]_0^1$$

$$=\frac{1}{4}-\frac{3}{2}+2$$

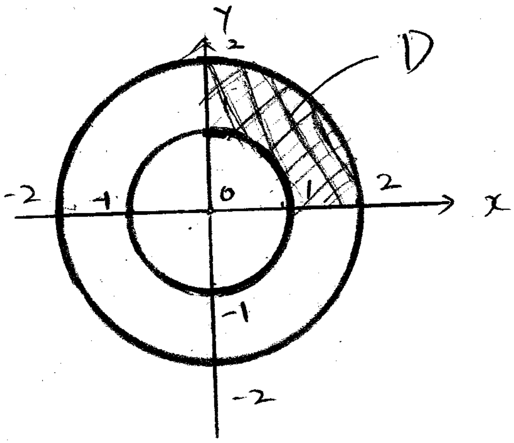
$$=\frac{1-6+8}{4}$$

$$=\frac{3}{4}$$

問4

(1)

$D = \{(x,y) \mid 1 \leq x^2+y^2 \leq 4,\ x \geq 0,\ y \geq 0\}$



(2)

$x(r,\theta) = r\cos\theta$

$y(r,\theta) = r\sin\theta$ として

$x_r = \dfrac{\partial x}{\partial r} = \cos\theta$,　$y_\theta = \dfrac{\partial y}{\partial \theta} = r\cos\theta$

(3)

$x_\theta = \dfrac{\partial x}{\partial \theta} = -r\sin\theta$,　$y_r = \dfrac{\partial y}{\partial r} = \sin\theta$

$$\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta$

$= r(\cos^2\theta + \sin^2\theta)$

$= r$

問4

(4)

$$\iint_D \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy \quad \cdots ①$$

$x = r\cos\theta$, $y = r\sin\theta$ として ① は.

(1), (2), (3) より

$$\iint_D \frac{r^2\cos^2\theta}{r} r\,dr\,d\theta$$

$$D = \{(r, \theta) \mid 1 \leqq r \leqq 2,\ 0 \leqq r \leqq \frac{\pi}{2}\}$$

$$\iint_D \frac{r^2\cos^2\theta}{r} r\,dr\,d\theta$$

$$= \int_1^2 \left\{ \int_0^{\frac{\pi}{2}} r^2\cos^2\theta\,d\theta \right\} dr$$

$$= \int_1^2 \left\{ \int_0^{\frac{\pi}{2}} r^2 \frac{1}{2}(\cos 2\theta + 1)\,d\theta \right\} dr$$

$$= \int_1^2 \left\{ \frac{1}{2} r^2 \left[\frac{1}{2}\sin 2\theta + \theta \right]_0^{\frac{\pi}{2}} \right\} dr$$

$$= \int_1^2 \frac{1}{2} r^2 \left\{ \left(\frac{1}{2}\sin\pi + \frac{\pi}{2} \right) - \left(\frac{1}{2}\sin 0 + 0 \right) \right\} dr$$

$$= \int_1^2 \frac{1}{2} r^2 \cdot \frac{\pi}{2}\,dr$$

$$= \frac{\pi}{4} \frac{1}{3} \left[r^3 \right]_1^2$$

$$= \frac{\pi}{12}(8-1) = \frac{7}{12}\pi$$

問5

(1)

$$y'' - 2y' + 2y = -x$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = -x \cdots ①$$

右辺の形から1つの解を

$y = Ax + B$ と予想する

(A, B は任意の定数)

$$\frac{dy}{dx} = A, \quad \frac{d^2y}{dx^2} = 0$$

①に代入して

$$-2A + 2(Ax + B) = -x$$

$$2Ax + (-2A + 2B) = -x$$

$$\begin{cases} 2A = -1 \\ -2A + 2B = 0 \end{cases}$$

$A = -\frac{1}{2}$ //

$B = -\frac{1}{2}$ // より1つの解は

$$y = -\frac{1}{2}x - \frac{1}{2}$$

次に $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$

の一般解を求める。

$y = Ce^{\alpha x}$ として

$$\frac{dy}{dx} = C\alpha e^{\alpha x}, \quad \frac{d^2y}{dx^2} = C\alpha^2 e^{\alpha x}$$

$$C\alpha^2 e^{\alpha x} - 2C\alpha e^{\alpha x} + 2Ce^{\alpha x} = 0$$

$$\alpha^2 - 2\alpha + 2 = 0$$

$$\alpha = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm 2i}{2}$$

$= 1 \pm i$ より、

$y = e^{x}(D\cos x + E\sin x)$ //

(C, D、E、α は任意の定数)

よって解は

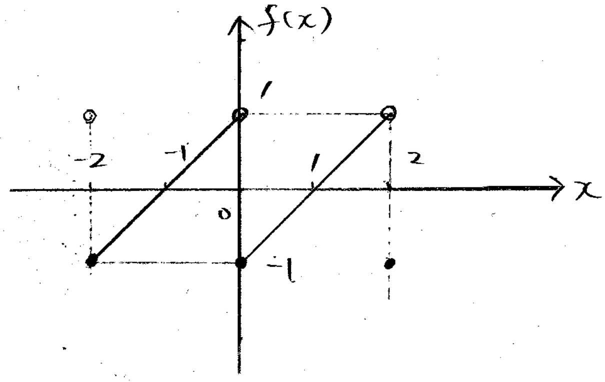
$y = -\frac{1}{2}x - \frac{1}{2} + e^{x}(D\cos x + E\sin x)$ //

問5

(2)

$f(x) = x - 1 \quad (0 \leq x < 2)$

※ $f(x)$は計算上奇関数のため
$a_0 = 0$, $a_n = 0$ として
考える



$2L = 2$

$L = 1$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

$$= \int_{-1}^{1} f(x)\sin\frac{n\pi}{L}x\,dx$$

$f(x)$は -1から1では
計算上は奇関数のため

$$= 2\int_0^1 (x-1)\sin n\pi x\,dx$$

$$= 2\left\{-\frac{1}{n\pi}\Big[(x-1)\cos n\pi x\Big]_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi x\,dx\right\}$$

$$= 2\left\{-\frac{1}{n\pi}(0+\cos 0) + \frac{1}{n\pi}\cdot\frac{1}{n\pi}\Big[\sin n\pi x\Big]_0^1\right\}$$

$$= 2\left\{-\frac{1}{n\pi} + \frac{1}{n^2\pi^2}(\sin n\pi - 0)\right\}$$

$$= -\frac{2}{n\pi}$$ よって $$f(x) \sim -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x)$$ //

問6

(1)

(i) $f(t) = 4 + (t^2 - 3te^t)e^{-2t}$

$$\mathcal{L}[f(t)] = \mathcal{L}[4 + (t^2 - 3te^t)e^{-2t}]$$

$$= \mathcal{L}[4] + \mathcal{L}[t^2 e^{-2t}] - \mathcal{L}[3te^{-t}]$$

$$= \frac{4}{s} + \frac{2!}{(s+2)^3} - 3\frac{1!}{(s+1)^2}$$

$$= \frac{4}{s} + \frac{2}{(s+2)^3} - \frac{3}{(s+1)^2}$$

(ii) $F(s) = \dfrac{s-1}{(s-3)^2}$

$$\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{s-1}{(s-3)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-3+2}{(s-3)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{s-3}{(s-3)^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s-3)^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s-3}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s-3)^2}\right]$$

$$= e^{3t} + 2te^{3t}$$

問6

(2)

$$\frac{d\vec{v}(t)}{dt} = \left(\cos 3t,\ \frac{1}{t+1},\ \sqrt{2t+3}\right)$$

$$\vec{v}(0) = (0,\ 0,\ 0)$$

$\vec{i}, \vec{j}, \vec{k}$ で書くと

$$\int \frac{d\vec{v}}{dt}\,dt = \int\left\{\cos 3t\,\vec{i} + \frac{1}{t+1}\vec{j} + \sqrt{2t+3}\,\vec{k}\right\}dt$$

$$= \frac{1}{3}\sin 3t\,\vec{i} + \log(t+1)\,\vec{j} + \frac{1}{3}(2t+3)^{\frac{3}{2}}\vec{k} + \vec{C}$$

($\vec{C}$は任意の定数ベクトル)

今、$\vec{v}(0) = \vec{0}$ より

$$\vec{v}(0) = \frac{1}{3}\sin 0\,\vec{i} + \log 1\,\vec{j} + \frac{1}{3}3\sqrt{3}\,\vec{k} + \vec{C} = \vec{0}$$

$$\sqrt{3}\,\vec{k} + \vec{C} = \vec{0}$$

$\vec{C} = -\sqrt{3}\,\vec{k}$ よって

$$\vec{v}(t) = \frac{1}{3}\sin 3t\,\vec{i} + \log(t+1)\,\vec{j} + \left(\frac{1}{3}(2t+3)^{\frac{3}{2}} - \sqrt{3}\right)\vec{k}$$

問6

(3) $\vec{A} = \left(\cos(x-y),\ ye^{2z},\ \log xz \right)$

$\vec{i}, \vec{j}, \vec{k}$ で書くと

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(x-y) & ye^{2z} & \log xz \end{vmatrix}$$

$$= \left(0 - 2ye^{2z} \right)\vec{i} - \left(\frac{z}{xz} - 0 \right)\vec{j} + \left(0 - \sin(x-y) \right)\vec{k}$$

$$= -2ye^{2z}\vec{i} - \frac{1}{x}\vec{j} - \sin(x-y)\vec{k}$$